Math Circles Grade 11/12 Session 1

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Matrices

$$\begin{array}{ccc} & \text{notebooks} & \text{pens} \\ \text{Anil} \rightarrow & \begin{bmatrix} 20 & 5 \end{bmatrix} & \end{array}$$

$$\text{Jake} \rightarrow \begin{bmatrix} 14 & 6 \end{bmatrix}$$

$$\text{Ala} \rightarrow \begin{bmatrix} 10 & 10 \end{bmatrix}$$

This information can be expressed in a matrix as follows:

$$\begin{array}{cl} \begin{bmatrix} 20 & 5 \\ 14 & 6 \\ 10 & 10 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \\ \leftarrow \text{Third row} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{First column} & \text{Second column} \end{array} & \end{array}$$

or

$$\begin{bmatrix} 20 & 14 & 10 \\ 5 & 6 & 10 \end{bmatrix}$$

Definition: A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

Matrices are denoted by capital letters.

Order of a matrix

A matrix having m rows and n columns is called a matrix of order $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$m \times n$: $m \to$ no. of rows, $n \to$ no. of columns

$$A = [a_{ij}]_{m \times n}, \quad 1 \le i \le m, \quad 1 \le j \le n, \quad i, j \in \mathbb{N}$$

$a_{ij} \to (i,j)^{th}$ element of A

Observe that the total number of elements in a matrix is the product of number of rows and number of columns of the matrix.

For the purpose of our sessions, we will assume that all entries of the matrices are real numbers.

Types of matrices

column matrix: order $\rightarrow m \times 1$, $m \in \mathbb{R}$

↳ One column only

eg. $A = \begin{bmatrix} 2 \\ 4 \\ 5.2 \end{bmatrix}$

row matrix: order $\rightarrow 1 \times n$, $n \in \mathbb{R}$

↳ One row only

square matrix: Order $\rightarrow n \times n$, $n \in \mathbb{R}$

number of rows = number of columns

eg. $A = \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix}$

diagonal matrix: (Only for square matrices)

eg. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

all entries except this diagonal must be zero.

scalar matrix: (Only for diagonal matrices)
All diagonal entries are equal.

eg. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

identity matrix: (Only for diagonal matrices)
All diagonal entries are 1.

eg. $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

zero matrix: All elements are zero.

eg. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Addition of matrices

$$\begin{bmatrix} 4 & 2 \\ 3 & -8 \end{bmatrix} + \begin{bmatrix} 12 & -3 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 4+12 & 2+(-3) \\ 3+5 & (-8)+(-1) \end{bmatrix}$$
$$= \begin{bmatrix} 16 & -1 \\ 8 & -9 \end{bmatrix}$$

For addition to be well-defined both matrices should be of the same order.

Multiplication of a matrix by a scaler

$$3A = 3\begin{bmatrix} 1 & -2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 3\times 1 & 3\times(-2) \\ 3\times 4 & 3\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -6 \\ 12 & 18 \end{bmatrix}$$

Negative of a matrix: $-A = (-1)A$

Difference of matrices: $A - B = A + (-1)B$

Multiplication of matrices

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix}_{2\times 2} \begin{bmatrix} 5 \\ 50 \end{bmatrix}_{2\times 1} = \begin{bmatrix} (2\times 5) + (5\times 50) \\ (8\times 5) + (10\times 50) \end{bmatrix}_{2\times 1} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}_{2\times 1}$$

$2\times 2 \quad 2\times 1$

Number of columns of first matrix must be equal to the number of rows of second matrix for the multiplication be well-defined.

The number of rows of the resulting matrix is equal to the number of rows of the first matrix.
The number of columns of the resulting matrix is equal to the number of columns of the second matrix .

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \times \begin{bmatrix} 5 & 4 \\ 50 & 40 \end{bmatrix} = \begin{bmatrix} (2\times5)+(5\times50) & (2\times4)+(5\times40) \\ (8\times5)+(10\times50) & (8\times4)+(10\times40) \end{bmatrix}$$

$$= \begin{bmatrix} 260 & 208 \\ 540 & 432 \end{bmatrix}$$

Note that if AB is defined that does not mean BA will be well-defined.

Also, $AB \neq BA$ for most cases.

eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

then $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

But,

for $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$,

$$AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$$

Transpose of a matrix

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & -2 & 5 \end{bmatrix}_{2\times3} \Longrightarrow A^T = \begin{bmatrix} 3 & 0 \\ -3 & -2 \\ 1 & 5 \end{bmatrix}_{3\times2}$$

Notation: A^T or A'

Properties of transpose of a matrix

(i) $(A+B)^T = A^T + B^T$

(ii) $(AB)^T = B^T A^T$

(iii) $(A^T)^T = A$

(iv) $(kA)^T = kA^T$ where $k \in \mathbb{R}$

$$(A+B)^T = \left(\begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\right)^T$$

$$= \begin{bmatrix} 1 & 6 \\ 0 & 4 \end{bmatrix}^T_{2\times 2}$$

$$= \begin{bmatrix} 1 & 0 \\ 6 & 4 \end{bmatrix}_{2\times 2}$$

$$A^T + B^T = \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix}^T + \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 6 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

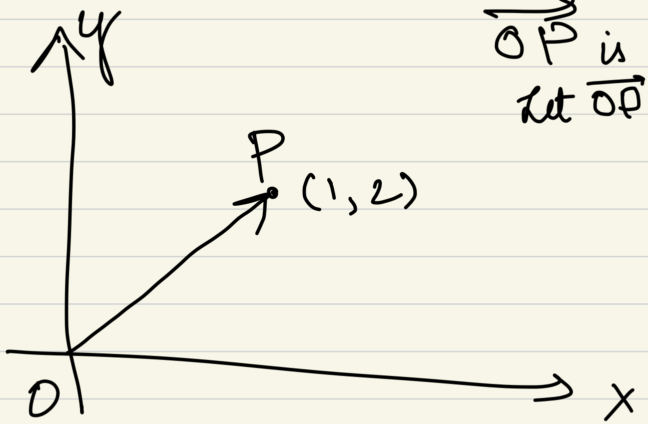
$$AB = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow (AB)^T = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Vectors — Column and row matrices



$\overrightarrow{OP}$ is a vector

Let $\overrightarrow{OP} = \vec{x}$.

Then $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\vec{x}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$W\vec{x}$ ⟶ We will need to check if such multiplications are well-defined when we do neural networks.

W ⟶ Weight matrix

$\vec{x}$ ⟶ vector

Session 1 questions

Q1. Consider a quadrilateral ABCD with vertices A(1,0), B(3,2), C(1,3) and D(-1,2). Represent this information in matrix form.

Q2- If a matrix has 8 elements, what are the possible orders it can have?

Q3- Construct a 3×2 matrix whose elements are given by $a_{ij} = i + j$.

Q4- Find the values of x, y, z for which $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$.

Q5- Find 2A-B if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$.

Q6- Find AB if $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$.

Q7- Find $(AB)^T$ for matrices in Q6.

Session 1 solutions

Sol.1- $$Q = \begin{array}{cccc} A & B & C & D \\ \end{array}$$
$$Q = \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2\times 4}$$

or
$$Q = \begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4\times 2}$$

Sol. 2- $1\times 8,\ 8\times 1,\ 2\times 4,\ 4\times 2$

Sol. 3- $$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$$

$a_{23} = 2+3$

Sol. 4- $x = 1,\ y = 4,\ z = 3$

Sol. 5- $$2A - B = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

Sol. 6- $$AB = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Sol. 7- $$(AB)^T = \begin{bmatrix} 75 & 25 \\ 117 & 39 \\ 72 & 24 \end{bmatrix}$$